SHARP WEIGHTED BOUNDS FOR MULTILINEAR MAXIMAL FUNCTIONS AND CALDERÓN-ZYGMUND OPERATORS

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ABSTRACT. In this paper we prove some sharp weighted norm inequalities for the multi(sub)linear maximal function $\mathcal{M}$ introduced in [18] and for multilinear Calderón-Zygmund operators. In particular we obtain a sharp mixed "$A_p - A_\infty$" bound for $\mathcal{M}$, some partial results related to a Buckley-type estimate for $\mathcal{M}$, and a sufficient condition for the boundedness of $\mathcal{M}$ between weighted L^p spaces with different weights taking into account the precise bounds.

Next we get a bound for multilinear Calderón-Zygmund operators in terms of dyadic positive multilinear operators in the spirit of the recent work [16]. Then we obtain a multilinear version of the "A_2 conjecture". Several open problems are posed.

1. INTRODUCTION

The modern theory of weighted norm inequalities for some of the main operators in Harmonic Analysis originated in the beginning of the 70's in the works by R. Hunt, B. Muckenhoupt, R. Wheeden, R. Coifman and C. Fefferman. In particular, it was realized that the key role in this theory is played by the so-called A_p condition. Much later, the question about the sharp dependence of the $L^p(w)$ operator norm in term of the the A_p constant or characteristic of the weight appeared. First, for the Hardy-Littlewood maximal operator this problem was solved by S. Buckley [2].

It turned out that for singular integrals the question is much more complicated. In [21], S. Petermichl solved it for the Hilbert transform. Recently, T. Hytönen [9] gave a complete solution for general Calderón-Zygmund operators solving the so-called A_2 conjecture. A bit later this result was improved in [12] in the case $p = 2$ and for general p in [10]. A further improvement was obtained in [11] where a non-probablistic proof was found together with a q-variation estimate. We refer to these papers for a more detailed history and for some other closely related results like (1.3).

The aim of this paper is to give some multilinear analogues of the above mentioned results in the spirit of the theory of multiple weights developed recently in [18]. We introduce some notation. Given $\vec{f} = (f_1, \ldots, f_m)$, we define the multi(sub)linear

maximal operator $\mathcal{M}$ by

$$\mathcal{M}(\vec{f})(x) = \sup_{Q \ni x} \prod_{i=1}^{m} \frac{1}{|Q|} \int_Q |f_i(y_i)| dy_i,$$

where the supremum is taken over all cubes Q containing x. It is shown in that paper that this operator controls in several ways the class of multilinear Calderón-Zygmund operators (see section 2.2). A particular instance of this intimate relationship is the class of weights characterizing the weighted L^p spaces for which both operators are bounded. To define this class of weights we let $\vec{w} = (w_1, \ldots, w_m)$ and $\vec{P} = (p_1, \ldots, p_m)$. Set $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $\nu_{\vec{w}} = \prod_{i=1}^{m} w_i^{p/p_i}$. We say that $\vec{w}$ satisfies the $A_{\vec{P}}$ condition if

$$[\vec{w}]_{A_{\vec{P}}} = \sup_Q \left(\frac{1}{|Q|} \int_Q \nu_{\vec{w}} \right) \prod_{i=1}^{m} \left(\frac{1}{|Q|} \int_Q w_i^{1-p_i'} \right)^{p/p_i'} < \infty.$$

It is easy to see that in the linear case (that is, if $m = 1$) $[\vec{w}]_{A_{\vec{P}}} = [w]_{A_p}$ is the usual A_p constant. In [18] the following multilinear extension of the Muckenhoupt A_p theorem for the maximal function was obtained: the inequality

$$(1.1) \qquad \|\mathcal{M}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \le C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i)}$$

holds for every $\vec{f}$ if and only if $\vec{w}$ satisfies the $A_{\vec{P}}$ condition.

The first question we are going to study is the question about the sharp dependence of C in (1.1). In the standard situation, namely when $m = 1$, two different types of sharp weighted inequalities for the Hardy-Littlewood maximal operator M are known. First, Buckley's theorem [2] says that for any $1 < p < \infty$,

$$(1.2) \qquad \|M\|_{L^p(w)} \le C_{n,p}[w]_{A_p}^{\frac{1}{p-1}}.$$

Until very recently this result was thought to be sharp since the exponent of $[w]_{A_p}$ cannot be improved. However, recently T. Hytönen and the third author [12] (see also [13] for another proof) showed that (1.2) can be improved in the following way. Define the A_∞ constant of w by

$$[w]_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q).$$

This definition goes back to the characterization of the A_∞ class of weights given by N. Fujii [7] (see also M. Wilson [22]). It is not difficult to show that $[w]_{A_\infty} \le c_n[w]_{A_r}$ for any $r \ge 1$. It was shown in [12] that

$$(1.3) \qquad \|M\|_{L^p(w)} \le C_{n,p}([w]_{A_p}[\sigma]_{A_\infty})^{1/p},$$

where $\sigma = w^{1-p'}$. This estimate implies (1.2) since $[\sigma]_{A_\infty} \le c[\sigma]_{A_{p'}} = c[w]_{A_p}^{\frac{1}{p-1}}$, and therefore $([w]_{A_p}[\sigma]_{A_\infty})^{1/p} \le c[w]_{A_p}^{\frac{1}{p-1}}$.

Our multilinear results related to (1.2) and (1.3) are somewhat surprising. They show that contrary to the case $m = 1$, multilinear versions of (1.2) and (1.3) are independent of each other. We are able to get a full analogue of (1.3). However such an analogue does not yield an expected full analogue of (1.2). Our first main result is the following.

Theorem 1.1. *Let $1 < p_i < \infty, i = 1, \ldots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}$. Then the inequality*

$$(1.4) \qquad \|\mathcal{M}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq C_{n,m,\vec{P}} [\vec{w}]_{A_{\vec{P}}}^{\frac{1}{p}} \prod_{i=1}^m ([\sigma_i]_{A_\infty})^{\frac{1}{p_i}} \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}$$

holds if $\vec{w} \in A_{\vec{P}}$, where $\sigma_i = w_i^{1-p_i'}$, $i = 1, \ldots, m$. Furthermore the exponents are sharp in the sense that they cannot be replaced by smaller ones.

Even though the result of this theorem is sharp it would be of interest to find an extension of Buckley's estimate (1.2). However this task seems to be more complicated. Currently we can get only several partial results expressed in the following theorem.

Theorem 1.2. *Let $1 < p_i < \infty, i = 1, \ldots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}$. Denote by $\alpha = \alpha(p_1, \ldots, p_m)$ the best possible power in*

$$(1.5) \qquad \|\mathcal{M}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq C_{n,m,p} [\vec{w}]_{A_{\vec{P}}}^{\alpha} \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}.$$

Then we have the following results:

(i) *for all $1 < p_1, \ldots, p_m < \infty$, $\frac{m}{mp-1} \leq \alpha \leq \frac{1}{p}\left(1 + \sum_{i=1}^m \frac{1}{p_i-1}\right)$;*

(ii) *if $p_1 = p_2 = \cdots = p_m = r > 1$, then $\alpha = \frac{m}{r-1}$.*

It is easy to see that the upper and lower bounds for α in (i) coincide if $m = 1$. The upper bound for α in (i) is a corollary of Theorem 1.1 after an application of Lemma 3.1 (see section 3). However, in the case of (ii) α coincides with the lower bound in (i). This says that if $m \geq 2$, then the upper bound in (i) is not sharp, in general. Hence, contrary to the linear situation, (1.4) cannot be used in order to get a sharp bound in terms of $[\vec{w}]_{A_{\vec{P}}}$. Perhaps, the explanation of this is that the right-hand side of (1.4) involves $m + 1$ independent suprema while the definition of $[\vec{w}]_{A_{\vec{P}}}$ involves only one supremum or else Lemma 3.1 is not sharp. Resuming, the problem of finding the sharp α in (1.5) remains open except the case considered in (ii).

We also give a sufficient condition for the "two-weighted" boundedness of $\mathcal{M}$ with precise bounds generalizing the corresponding linear result from [20] and its multilinear counterpart in [19]. Let X be a Banach function space. By X' we denote the associate space to X. Given a cube Q, define the X-average of f over Q and

the maximal operator M_X by

$$\|f\|_{X,Q} = \|\tau_{\ell_Q}(f\chi_Q)\|_X, \quad M_X f(x) = \sup_{Q \ni x} \|f\|_{X,Q},$$

where ℓ_Q denotes the side length of Q and where $\tau_\delta f = f(\delta x),\ \delta > 0, x \in \mathbb{R}^n.$

Theorem 1.3. *Let* $1 < p_i < \infty, i = 1, \ldots, m$ *and* $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}.$ *Let* X_i *be a Banach function space such that* $M_{X_i'}$ *is bounded on* $L^{p_i}(\mathbb{R}^n).$ *Let* u *and* $v_1, \ldots, v_m$ *be the weights satisfying*

$$K = \sup_Q \Big(\frac{u(Q)}{|Q|}\Big)^{\frac{1}{p}} \prod_{i=1}^m \|v_i^{-1/p_i}\|_{X_i,Q} < \infty.$$

Then

$$\|\mathcal{M}(\vec{f})\|_{L^p(u)} \le C_{n,m} K \prod_{i=1}^m \|M_{X_i'}\|_{L^{p_i}(\mathbb{R}^n)} \|f_i\|_{L^{p_i}(v_i)}.$$

This result can be seen as a two weight version of (1.4) when considering function spaces X given by $X = L^{rp'}$ for $1 < p, r < \infty$ so that

$$\|M_{X'}\|_{L^p(\mathbb{R}^n)} = \|M_{(rp')'}\|_{L^p(\mathbb{R}^n)} \approx (r')^{1/p}.$$

Another interesting example is given when considering the Orlicz space space $X = L_B$ where B is a Young function for which $\|M_{X'}\|_{L^p(\mathbb{R}^n)} = \|M_{\bar{B}}\|_{L^p(\mathbb{R}^n)}$ is finite. In particular if $B(t) = t^{p'} (\log(e+t))^{p'-1+\delta},\ \delta > 0, 1 < p < \infty$ it follows from [20] that

$$\|M_{X'}\|_{L^p(\mathbb{R}^n)} = \|M_{\bar{B}}\|_{L^p(\mathbb{R}^n)} \approx (\frac{1}{\delta})^{1/p}.$$

We turn now to some multilinear analogues of the sharp weighted results for singular integrals. First, we recall some linear results. Let T be a Calderón-Zygmund operator. Then it was proved by T. Hytönen [9] in full generality that

$$(1.6) \qquad \|T\|_{L^p(w)} \le C_{T,n,p}[w]_{A_p}^{\max(1,\frac{1}{p-1})} \quad (1 < p < \infty).$$

Observe that it suffices to prove (1.6) in the case $p = 2$; then for any other p the result follows by the sharp extrapolation theorem. Note also that (1.6) in the case $p = 2$ was usually referred as the A_2 conjecture. In two recent papers [16, 17] a different proof of (1.6) was found by the second author. This proof shows that there is an intimate relationship between the (continuous) singular integral T and some very special dyadic type operators. These operators are defined by means of the concept of sparseness. Given a sparse family $\mathcal{S} = \{Q_j^k\}$ of cubes from a dyadic grid $\mathscr{D}$, (these notions will be defined in Section 2.1) we consider the operator $\mathcal{A}_{\mathcal{S},\mathscr{D}}$ defined by

$$\mathcal{A}_{\mathcal{S},\mathscr{D}}f(x) = \sum_{j,k} f_{Q_j^k}\chi_{Q_j^k}(x).$$

It was proved in [16, 17] that for any Banach function space X,

$$(1.7) \qquad \|Tf\|_X \le c_{T,n} \sup_{\mathcal{S},\mathscr{D}} \|\mathcal{A}_{\mathcal{S},\mathscr{D}}|f|\|_X.$$

A rather simple argument found in [3] shows that $\|\mathcal{A}_{\mathcal{S},\mathscr{D}}\|_{L^2(w)} \le c[w]_{A_2}$. Therefore if $X = L^2(w)$, then (1.7) implies (1.6) for $p = 2$.

The second main result of this paper is an extension of (1.7) to the multilinear setting. Here, the dyadic operator that appears naturally is a multilinear version of the operator $\mathcal{A}_{\mathscr{D},\mathcal{S}}$ given by

$$\mathcal{A}_{\mathscr{D},\mathcal{S}}(\vec{f})(x) = \sum_{j,k} \Big(\prod_{i=1}^m (f_i)_{Q_j^k} \Big) \chi_{Q_j^k}(x).$$

Given $\vec{f} = (f_1, \ldots, f_m)$, denote $|\vec{f}| = (|f_1|, \ldots, |f_m|)$.

Theorem 1.4. *Let $T(\vec{f})$ be a multilinear Calderón-Zygmund operator and let X be a Banach function space over $\mathbb{R}^n$ equipped with Lebesgue measure. Then, for any appropriate $\vec{f}$,*

$$\|T(\vec{f})\|_X \le c_{T,m,n} \sup_{\mathscr{D},\mathcal{S}} \|\mathcal{A}_{\mathscr{D},\mathcal{S}}(|\vec{f}|)\|_X,$$

where the supremum is taken over arbitrary dyadic grids $\mathscr{D}$ and sparse families $\mathcal{S} \in \mathscr{D}$.

Similarly to the linear case, we would like to apply this result when $X = L^p(\nu_{\vec{w}})$. However, the exponent p is allowed to be smaller than one, to be more precise $1/m < p < \infty$ (see section 2.2). Therefore, if $1/m < p < 1$, Theorem 1.4 cannot be applied since in this case the space $L^p(\nu_{\vec{w}})$ is not a Banach space. This raises an interesting question whether the condition in Theorem 1.4 that X is a Banach space can be relaxed until that X is a quasi-Banach space. It is natural to consider first this question in the linear situation. Observe that in the current proof of (1.7) the fact that X is a Banach space was essential.

The third main result of this paper can be seen as a multilinear version of the A_2 conjecture (1.6). Indeed, one of the main results obtained in [18] is that if $\vec{w} \in A_{\vec{P}}$, then an analogue of (1.1) holds with $T(\vec{f})$ instead of $\mathcal{M}(\vec{f})$. Hence, the question about the sharp dependence on $[\vec{w}]_{A_{\vec{P}}}$ in the corresponding inequality is quite natural. It is interesting that contrary to the linear situation where (1.7) implies the A_2 conjecture, we are currently able to apply Theorem 1.4 only in one particular case being the content of the following theorem.

Theorem 1.5. *Let $T(\vec{f})$ be a multilinear Calderón-Zygmund operator. Assume that $p_1 = p_2 = \cdots = p_m = m + 1$. Then*

$$(1.8) \qquad \|T(\vec{f})\|_{L^p(\nu_{\vec{w}})} \le C_{T,m,n}[\vec{w}]_{A_{\vec{P}}} \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}.$$

Several remarks about this result are in order. As we already mentioned, Theorem 1.5 can be regarded as a multilinear "A_2 conjecture". However, it is natural to ask how to extend it to all $1 < p_i < \infty$. This leads to several interesting problems. As we explained above, (1.6) can be obtained from the case $p = 2$ by the sharp version of the extrapolation theorem of Rubio de Francia obtained in [5] (see also [6] or [4] for different and simpler proofs). It would be very desirable to get a multilinear analogue of this result. Having such an analogue, inequality (1.8) probably would be a starting point to extrapolate from.

Observe, however, that (1.6) can be proved also without the use of extrapolation. Indeed, one can easily prove (1.6) with $\mathcal{A}_{\mathscr{D},\mathcal{S}}$ instead of T for all $1 < p < \infty$ (as it was done in [3] for $p = 2$), and then apply (1.7) with $X = L^p(w)$. The proof of (1.6) for $\mathcal{A}_{\mathscr{D},\mathcal{S}}$ is very close in spirit to the proof of Buckley's inequality (1.2) found in [14]. Hence, it is natural to ask whether it is possible to find a similar proof for a multilinear version of $\mathcal{A}_{\mathscr{D},\mathcal{S}}$. But this leads to a problem of finding α in Theorem 1.2 by the method in [14]. Part (ii) of Theorem 1.2 indeed obtained by an adaptation of this method. However how to do that in the case of different p_i is not clear.

The paper is organized as follows. Some preliminaries are contained in Section 2. In Section 3 we prove all theorems related to $\mathcal{M}$. Section 4 is devoted to the proof of Theorem 1.4. Finally, Theorem 1.5 is proved in Section 5.

2. Preliminaries

2.1. **Dyadic grids.** Recall that the standard dyadic grid in $\mathbb{R}^n$ consists of the cubes

$$2^{-k}([0,1)^n + j), \quad k \in \mathbb{Z}, j \in \mathbb{Z}^n.$$

Denote the standard grid by $\mathcal{D}$.

By a *general dyadic grid* $\mathscr{D}$ we mean a collection of cubes with the following properties: (i) for any $Q \in \mathscr{D}$ its sidelength ℓ_Q is of the form $2^k, k \in \mathbb{Z}$; (ii) $Q \cap R \in \{Q, R, \emptyset\}$ for any $Q, R \in \mathscr{D}$; (iii) the cubes of a fixed sidelength 2^k form a partition of $\mathbb{R}^n$.

We say that $\{Q_j^k\}$ is a *sparse family* of cubes if: (i) the cubes Q_j^k are disjoint in j, with k fixed; (ii) if $\Omega_k = \cup_j Q_j^k$, then $\Omega_{k+1} \subset \Omega_k$; (iii) $|\Omega_{k+1} \cap Q_j^k| \le \frac{1}{2}|Q_j^k|$.

With each sparse family $\{Q_j^k\}$ we associate the sets $E_j^k = Q_j^k \setminus \Omega_{k+1}$. Observe that the sets E_j^k are pairwise disjoint and $|Q_j^k| \le 2|E_j^k|$.

Given a cube Q_0, denote by $\mathcal{D}(Q_0)$ the set of all dyadic cubes with respect to Q_0, that is, the cubes from $\mathcal{D}(Q_0)$ are formed by repeated subdivision of Q_0 and each of its descendants into 2^n congruent subcubes. Observe that if $Q_0 \in \mathscr{D}$, then each cube from $\mathcal{D}(Q_0)$ will also belong to $\mathscr{D}$.

We will use the following proposition from [12].

Proposition 2.1. *There are 2^n dyadic grids $\mathscr{D}_\alpha$ such that for any cube $Q \subset \mathbb{R}^n$ there exists a cube $Q_\alpha \in \mathscr{D}_\alpha$ such that $Q \subset Q_\alpha$ and $\ell_{Q_\alpha} \le 6\ell_Q$.*

Lemma 2.2. *For any non-negative integrable $f_i, i = 1, \ldots, m$, there exist sparse families $\mathcal{S}_\alpha \in \mathscr{D}_\alpha$ such that for all $x \in \mathbb{R}^n$,*

$$\mathcal{M}(\vec{f})(x) \leq (2 \cdot 12^n)^m \sum_{\alpha=1}^{2^n} \mathcal{A}_{\mathscr{D}_\alpha, \mathcal{S}_\alpha}(\vec{f})(x).$$

Proof. First, by Proposition 2.1,

$$(2.1) \qquad\qquad \mathcal{M}(\vec{f})(x) \leq 6^{mn} \sum_{\alpha=1}^{2^n} \mathcal{M}^{\mathscr{D}_\alpha}(\vec{f})(x).$$

Consider $\mathcal{M}^d(\vec{f})$ taken with respect to the standard dyadic grid. We will use exactly the same argument as in the Calderón-Zygmund decomposition. For c_n which will be specified below and for $k \in \mathbb{Z}$ consider the sets

$$\Omega_k = \{x \in \mathbb{R}^n : \mathcal{M}^d(\vec{f})(x) > c_n^k\}.$$

Then we have that $\Omega_k = \cup_j Q_j^k$, where the cubes Q_j^k are pairwise disjoint with k fixed, and

$$c_n^k < \prod_{i=1}^m (f_i)_{Q_j^k} \leq 2^{mn} c_n^k.$$

From this and from Hölder's inequality,

$$
\begin{aligned}
|Q_j^k \cap \Omega_{k+1}| &= \sum_{Q_l^{k+1} \subset Q_j^k} |Q_l^{k+1}| \\
&< c_n^{-\frac{k+1}{m}} \sum_{Q_l^{k+1} \subset Q_j^k} \prod_{i=1}^m \left(\int_{Q_l^{k+1}} f_i \right)^{1/m} \\
&\leq c_n^{-\frac{k+1}{m}} \prod_{i=1}^m \left(\int_{Q_j^k} f_i \right)^{1/m} \leq 2^n c_n^{-1/m} |Q_j^k|.
\end{aligned}
$$

Hence, taking $c_n = 2^{m(n+1)}$, we obtain that the family $\{Q_j^k\}$ is sparse, and

$$\mathcal{M}^d(\vec{f})(x) \leq 2^{m(n+1)} \mathcal{A}_{\mathcal{D}, \mathcal{S}}(\vec{f})(x).$$

Applying the same argument to each $\mathcal{M}^{\mathscr{D}_\alpha}(\vec{f})$ and using (2.1), we get the statement of the lemma. $\qquad\square$

2.2. **Multilinear Calderón-Zygmund operators.** Let T be a multilinear operator initially defined on the m-fold product of Schwartz spaces and taking values into the space of tempered distributions,

$$T : S(\mathbb{R}^n) \times \cdots \times S(\mathbb{R}^n) \to S'(\mathbb{R}^n).$$

We say that T is an m-linear Calderón-Zygmund operator if, for some $1 \leq q_j < \infty$, it extends to a bounded multilinear operator from $L^{q_1} \times \cdots \times L^{q_m}$ to L^q, where

$\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, and if there exists a function K, defined off the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfying

$$T(f_1, \ldots, f_m)(x) = \int\limits_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) f_1(y_1) \ldots f_m(y_m) \, dy_1 \ldots dy_m$$

for all $x \notin \cap_{j=1}^m \operatorname{supp} f_j$,

$$|K(y_0, y_1, \ldots, y_m)| \leq \frac{A}{\left(\sum\limits_{k,l=0}^m |y_k - y_l|\right)^{mn}},$$

and

$$|K(y_0, \ldots, y_j, \ldots, y_m) - K(y_0, \ldots, y_j', \ldots, y_m)| \leq \frac{A|y_j - y_j'|^\epsilon}{\left(\sum\limits_{k,l=0}^m |y_k - y_l|\right)^{mn+\epsilon}},$$

for some $\epsilon > 0$ and all $0 \leq j \leq m$, whenever $|y_j - y_j'| \leq \frac{1}{2} \max\limits_{0 \leq k \leq m} |y_j - y_k|$.

It was shown in [8] that if $\frac{1}{r_1} + \cdots + \frac{1}{r_m} = \frac{1}{r}$, then an m-linear Calderón-Zygmund operator satisfies

$$T : L^{r_1}(\mathbb{R}^n) \times \cdots \times L^{r_m}(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$$

when $1 < r_j < \infty$ for all $j = 1, \cdots, m$. Similarly if $1 \leq r_j \leq \infty$ for all $j = 1, \cdots, m$, we have

(2.2) $$T : L^1(\mathbb{R}^n) \times \cdots \times L^1(\mathbb{R}^n) \to L^{1/m,\infty}(\mathbb{R}^n)$$

2.3. **A "local mean oscillation decomposition".** The non-increasing rearrangement of a measurable function f on $\mathbb{R}^n$ is defined by

$$f^*(t) = \inf\{\alpha > 0 : |\{x \in \mathbb{R}^n : |f(x)| < \alpha\}| < t\} \quad (0 < t < \infty).$$

Given a measurable function f on $\mathbb{R}^n$ and a cube Q, the local mean oscillation of f on Q is defined by

$$\omega_\lambda(f; Q) = \inf_{c \in \mathbb{R}} \left((f - c)\chi_Q\right)^* (\lambda|Q|) \quad (0 < \lambda < 1).$$

By a median value of f over Q we mean a possibly nonunique, real number $m_f(Q)$ such that

$$\max\left(|\{x \in Q : f(x) > m_f(Q)\}|, |\{x \in Q : f(x) < m_f(Q)\}|\right) \leq |Q|/2.$$

It is easy to see that the set of all median values of f is either one point or the closed interval. In the latter case we will assume for the definiteness that $m_f(Q)$ is the *maximal* median value. Observe that it follows from the definitions that

(2.3) $$|m_f(Q)| \leq (f\chi_Q)^*(|Q|/2).$$

Given a cube Q_0, the dyadic local sharp maximal function $m^{\#,d}_{\lambda;Q_0} f$ is defined by

$$m^{\#,d}_{\lambda;Q_0} f(x) = \sup_{x \in Q' \in \mathcal{D}(Q_0)} \omega_\lambda(f; Q').$$

The following theorem was proved in [16] (its very similar version can be found in [15]).

Theorem 2.3. *Let f be a measurable function on $\mathbb{R}^n$ and let Q_0 be a fixed cube. Then there exists a (possibly empty) sparse family of cubes $Q^k_j \in \mathcal{D}(Q_0)$ such that for a.e. $x \in Q_0$,*

$$|f(x) - m_f(Q_0)| \le 4 m^{\#,d}_{\frac{1}{2^{n+2}};Q_0} f(x) + 2 \sum_{k,j} \omega_{\frac{1}{2^{n+2}}}(f; Q^k_j) \chi_{Q^k_j}(x).$$

2.4. **Banach function spaces.** For a general account of Banach function spaces we refer to [1, Ch. 1]. We mention only several notions which will be used below.

The associate space X' consists of measurable functions f for which

$$\|f\|_{X'} = \sup_{\|g\|_X \le 1} \int_{\mathbb{R}^n} |f(x)g(x)| dx < \infty.$$

This definition implies the following Hölder inequality:

(2.4) $$\int_{\mathbb{R}^n} |f(x)g(x)| dx \le \|f\|_X \|g\|_{X'}.$$

Further [1, p. 13],

(2.5) $$\|f\|_X = \sup_{\|g\|_{X'}=1} \int_{\mathbb{R}^n} |f(x)g(x)| dx.$$

3. Proof of Theorems 1.1, 1.2 and 1.3

In the proof of Theorem 1.1 we shall use the following reverse Hölder property of A_∞ weights proved in [12]: if $w \in A_\infty$, then

(3.1) $$\left(\frac{1}{|Q|} \int_Q w^{r(w)} \right)^{1/r(w)} \le 2 \frac{1}{|Q|} \int_Q w,$$

where $r(w) = 1 + \frac{1}{\tau_n [w]_{A_\infty}}$ and $\tau_n = 2^{11+n}$. Observe that $r'(w) \approx [w]_{A_\infty}$.

Proof of Theorem 1.1. By (2.1), it suffices to prove the theorem for the dyadic maximal operators $\mathcal{M}^{\mathscr{D}_\alpha}$. Since the proof is independent of the particular dyadic grid, without loss of generality we consider $\mathcal{M}^d$ taken with respect to the standard dyadic grid $\mathcal{D}$.

Let $a = 2^{m(n+1)}$. and $\Omega_k = \{x \in \mathbb{R}^n : \mathcal{M}^d(\vec{f})(x) > a^k\}$. We have seen in the proof of Lemma 2.2 that $\Omega_k = \cup_j Q^k_j$, where the family $\{Q^k_j\}$ is sparse and

$a^k < \prod_{i=1}^m \frac{1}{|Q_j^k|} \int_{Q_j^k} |f_i| \leq 2^{nm} a^k$. It follows that

$$\int_{\mathbb{R}^n} \mathcal{M}^d(\vec{f})^p \, \nu_{\vec{w}} dx = \sum_k \int_{\Omega_k \setminus \Omega_{k+1}} \mathcal{M}^d(\vec{f})^p \, \nu_{\vec{w}} dx$$

$$\leq a^p \sum_{k,j} \left(\prod_{i=1}^m \frac{1}{|Q_j^k|} \int_{Q_j^k} |f_i| dy_i \right)^p \nu_{\vec{w}}(Q_j^k)$$

$$\leq a^p \sum_{k,j} \left(\prod_{i=1}^m \frac{1}{|Q_j^k|} \int_{Q_j^k} |f_i| w_i^{\frac{1}{p_i}} w_i^{-\frac{1}{p_i}} dy_i \right)^p \nu_{\vec{w}}(Q_j^k)$$

$$\leq a^p \sum_{k,j} \prod_{i=1}^m \left(\frac{1}{|Q_j^k|} \int_{Q_j^k} |f_i|^{\alpha_i} w_i^{\frac{\alpha_i}{p_i}} dy_i \right)^{\frac{p}{\alpha_i}} \left(\frac{1}{|Q_j^k|} \int_{Q_j^k} w_i^{-\frac{\alpha_i'}{p_i}} dy_i \right)^{\frac{p}{\alpha_i'}} \nu_{\vec{w}}(Q_j^k),$$

where $\alpha_i = (p_i' r_i)'$ and r_i is the exponent in the sharp reverse Hölder inequality (3.1) for the weights σ_i which are in A_∞ for $i = 1, \ldots, m$. Applying (3.1) for each σ_i, we obtain

$$\int_{\mathbb{R}^n} \mathcal{M}^d(\vec{f})^p \, \nu_{\vec{w}} dx \leq a^p \sum_{k,j} \prod_{i=1}^m \left(\frac{1}{|Q_j^k|} \int_{Q_j^k} |f_i|^{\alpha_i} w_i^{\frac{\alpha_i}{p_i}} dy_i \right)^{\frac{p}{\alpha_i}}$$

$$\times \left(2 \frac{1}{|Q_j^k|} \int_{Q_j^k} \sigma_i \right)^{\frac{p}{p_i'}} \nu_{\vec{w}}(Q_j^k)$$

$$\leq C[\vec{w}]_{A_{\vec{P}}} \sum_{k,j} \prod_{i=1}^m \left(\frac{1}{|Q_j^k|} \int_{Q_j^k} |f_i|^{\alpha_i} w_i^{\frac{\alpha_i}{p_i}} dy_i \right)^{\frac{p}{\alpha_i}} |Q_j^k|.$$

Let E_j^k be the sets associated with the family $\{Q_j^k\}$. Using the properties of E_j^k and Hölder's inequality with the exponents p_i/p, we get

$$\int_{\mathbb{R}^n} \mathcal{M}^d(\vec{f})^p \, \nu_{\vec{w}} dx \leq 2C[\vec{w}]_{A_{\vec{P}}} \sum_{k,j} \prod_{i=1}^m \left(\frac{1}{|Q_j^k|} \int_{Q_j^k} |f_i(y_i)|^{\alpha_i} w_i^{\frac{\alpha_i}{p_i}} dy_i \right)^{\frac{p}{\alpha_i}} |E_j^k|$$

$$\leq 2C[\vec{w}]_{A_{\vec{P}}} \sum_{k,j} \int_{E_j^k} \prod_{i=1}^m M\left(|f_i|^{\alpha_i} w_i^{\frac{\alpha_i}{p_i}} \right)^{\frac{p}{\alpha_i}} dx$$

$$\leq 2C[\vec{w}]_{A_{\vec{P}}} \int_{\mathbb{R}^n} \prod_{i=1}^m M\left(|f_i|^{\alpha_i} w_i^{\frac{\alpha_i}{p_i}} \right)^{\frac{p}{\alpha_i}} dx$$

$$\leq 2C[\vec{w}]_{A_{\vec{P}}} \prod_{i=1}^m \left(\int_{\mathbb{R}^n} M\left(|f_i|^{\alpha_i} w_i^{\frac{\alpha_i}{p_i}} \right)^{\frac{p_i}{\alpha_i}} dx \right)^{\frac{p}{p_i}}.$$

From this and by the boundedness of M,

$$\int_{\mathbb{R}^n} \mathcal{M}^d(\vec{f})^p \, \nu_{\vec{w}} dx \leq C[\vec{w}]_{A_{\vec{P}}} \prod_{i=1}^m \left((p_i/\alpha_i)'\right)^{\frac{p}{p_i}} \left\| |f_i|^{\alpha_i} w_i^{\frac{\alpha_i}{p_i}} \right\|_{L^{\frac{p_i}{\alpha_i}}(\mathbb{R}^n)}^{\frac{p}{\alpha_i}}$$

$$\leq C[\vec{w}]_{A_{\vec{P}}} \prod_{i=1}^m (p_i' r_i')^{\frac{p}{p_i}} \|f_i\|_{L^{p_i}(w_i)}^p$$

$$\leq C[\vec{w}]_{A_{\vec{P}}} \prod_{i=1}^m ([\sigma_i]_{A_\infty})^{\frac{p}{p_i}} \|f_i\|_{L^{p_i}(w_i)}^p,$$

where in next to last inequality we have used that $(p_i/\alpha_i)' \leq p_i' r_i'$ and in the last inequality we have used that $r_i' \approx [\sigma_i]_{A_\infty}$, for $i = 1, \ldots, m$. This completes the proof of (1.4).

Let us show now the sharpness of the exponents in this inequality. Assume that $n = 1$ and $0 < \varepsilon < 1$. Let

$$w_i(x) = |x|^{(1-\varepsilon)(p_i-1)} \quad \text{and} \quad f_i(x) = x^{-1+\varepsilon} \chi_{(0,1)}(x), \quad i = 1, \ldots, m.$$

It is easy to check that $\nu_{\vec{w}} = |x|^{(1-\varepsilon)(pm-1)}$,

$$(3.2) \qquad [\vec{w}]_{A_{\vec{P}}} = [\nu_{\vec{w}}]_{A_{pm}} \approx (1/\varepsilon)^{mp-1} \quad \text{and} \quad [\sigma_i]_{A_\infty} \leq \frac{C}{\varepsilon}.$$

Also,

$$(3.3) \qquad \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)} = (1/\varepsilon)^{1/p}.$$

Let $f = x^{-1+\varepsilon} \chi_{(0,1)}(x)$. Then the left-hand side of (1.4) can be bounded from below as follows:

$$(3.4) \qquad \|\mathcal{M}(\vec{f})\|_{L^p(\nu_{\vec{w}})} = \|Mf\|_{L^{pm}(\nu_{\vec{w}})}^m \geq (1/\varepsilon)^m \|f\|_{L^{pm}(\nu_{\vec{w}})}^m = (1/\varepsilon)^{m+1/p}.$$

On the other hand, by (3.2) and (3.3), the right-hand side of (1.4) is at most $(1/\varepsilon)^{m+1/p}$. Since ε is arbitrary, this shows that the exponents $1/p$ and $1/p_i$ on the right-hand side of (1.4) cannot be replaced by smaller ones. $\square$

In order to get an upper bound for α in part (ii) of Theorem 1.2, we shall need the following technical lemma. Its proof follows the same lines as the proof of [18, Th. 3.6].

Lemma 3.1. *Let* $1 < p_j < \infty, j = 1, \ldots, m$ *and* $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}$. *If* $\vec{w} \in A_{\vec{P}}$, *then*

$$[\sigma_j]_{A_\infty} \leq C[\vec{w}]_{A_{\vec{P}}}^{p_j'/p}.$$

Proof. It was shown in [18, Th. 3.6] that if $\vec{w} \in A_{\vec{P}}$, then $\sigma_j \in A_{mp_j'}$. Our goal now is to check that

$$(3.5) \qquad [\sigma_j]_{A_{mp_j'}} \leq [\vec{w}]_{A_{\vec{P}}}^{p_j'/p}.$$

Since $[\sigma_j]_{A_\infty} \le C[\sigma_j]_{A_{mp'_j}}$, (3.5) would imply the statement of the lemma.

Fix $1 \le j \le m$, and define the numbers

$$q_j = p\Big(m - 1 + \frac{1}{p_j}\Big) \quad \text{and} \quad q_i = \frac{p_i}{p_i - 1}\frac{q_j}{p}, i \ne j.$$

Since

$$\sum_{i=1}^{m} \frac{1}{q_i} = \frac{1}{m - 1 + 1/p_j}\Big(\frac{1}{p} + \sum_{i=1,i\ne j}^{m}(1 - 1/p_i)\Big) = 1,$$

using Hölder inequality, we obtain

$$\int_Q w_j^{\frac{p}{p_j q_j}} = \int_Q \Big(\prod_{i=1}^{m} w_i^{\frac{p}{p_i q_j}}\Big)\Big(\prod_{i=1,i\ne j}^{m} w_i^{-\frac{p}{p_i q_j}}\Big)$$

$$\le \Big(\int_Q \prod_{i=1}^{m} w_i^{p/p_i}\Big)^{1/q_j} \prod_{i=1,i\ne j}^{m}\Big(\int_Q w_i^{-1/(p_i-1)}\Big)^{1/q_i}.$$

From this,

$$\Big(\int_Q w_j^{1-p'_j}\Big)\Big(\int_Q w_j^{\frac{p}{p_j q_j}}\Big)^{\frac{q_j p_j}{p(p_j-1)}}$$

$$\le \Big(\int_Q w_j^{1-p'_j}\Big)\Bigg[\Big(\int_Q \prod_{i=1}^{m} w_i^{p/p_i}\Big)^{1/q_j} \prod_{i=1,i\ne j}^{m}\Big(\int_Q w_i^{1-p'_i}\Big)^{1/q_i}\Bigg]^{\frac{q_j p_j}{p(p_j-1)}}$$

$$\le \Big(\int_Q w_j^{1-p'_j}\Big)\Bigg[\Big(\int_Q \prod_{i=1}^{m} w_i^{p/p_i}\Big)\prod_{i=1,i\ne j}^{m}\Big(\int_Q w_i^{1-p'_i}\Big)^{q_j/q_i}\Bigg]^{\frac{p'_j}{p}}.$$

Since

$$\frac{q_j}{q_i} = \frac{p\Big(m - 1 + \frac{1}{p_j}\Big)}{\frac{p_i}{p_i-1}\frac{p\big(m-1+\frac{1}{p_j}\big)}{p}} = \frac{p}{p'_i},$$

we obtain

$$\Big(\int_Q w_j^{1-p'_j}\Big)\Big(\int_Q w_j^{\frac{p}{p_j q_j}}\Big)^{\frac{q_j p_j}{p(p_j-1)}}$$

$$\le \Big(\int_Q w_j^{1-p'_j}\Big)\Bigg[\Big(\int_Q \prod_{i=1}^{m} w_i^{p/p_i}\Big)\prod_{i=1,i\ne j}^{m}\Big(\int_Q w_i^{1-p'_i}\Big)^{p/p'_i}\Bigg]^{\frac{p'_j}{p}}$$

$$\le \Bigg[\Big(\int_Q \prod_{i=1}^{m} w_i^{p/p_i}\Big)\prod_{i=1}^{m}\Big(\int_Q w_i^{1-p'_i}\Big)^{p/p'_i}\Bigg]^{\frac{p'_j}{p}}.$$

Therefore,

$$
\begin{aligned}
A_{mp'_j}(\sigma_j; Q) &= \left(\frac{1}{|Q|}\int_Q w_j^{1-p'_j}\right)\left(\frac{1}{|Q|}\int_Q w_j^{\frac{p}{p_j q_j}}\right)^{\frac{q_j p_j}{p(p_j-1)}} \\
&\leq \left[\left(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}\right)\prod_{i=1}^{m}\left(\frac{1}{|Q|}\int_Q w_i^{1-p'_i}\right)^{p/p'_i}\right]^{\frac{p'_j}{p}} \\
&= (A_{\vec{P}}(\vec{w}; Q))^{p'_j/p},
\end{aligned}
$$

which proves (3.5). $\qquad\square$

Proof of Theorem 1.2. We start with part (i). Consider the example given in the proof of Theorem 1.1. Combining (3.2), (3.3) and (3.4) with

$$
\|\mathcal{M}(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leq C[\vec{w}]_{A_{\vec{P}}}^{\alpha}\prod_{j=1}^{m}\|f_j\|_{L^{p_j}(w_j)},
$$

we obtain $m + 1/p \leq \alpha(mp-1) + 1/p$ which yields $\alpha \geq \frac{m}{mp-1}$.

Further, by Theorem 1.1 and Lemma 3.1,

$$
\alpha \leq \frac{1}{p} + \sum_{i=1}^{m}\frac{1}{p_i}\frac{p'_i}{p} = \frac{1}{p}\left(1 + \sum_{i=1}^{m}\frac{1}{p_i-1}\right).
$$

This completes the proof of part (i).

Suppose now that $p_1 = p_2 = \cdots = p_m = r$. Then $p = r/m$, $\nu_{\vec{w}} = \left(\prod_{j=1}^{m} w_j\right)^{1/m}$. Denote

$$
A_{\vec{P}}(\vec{w}; Q) = \left(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}\right)\prod_{i=1}^{m}\left(\frac{1}{|Q|}\int_Q \sigma_i\right)^{(r-1)/m},
$$

where $\sigma_i = w_i^{1-r'}$. Set also

$$
\mathcal{M}_{\vec{\sigma}}(\vec{f})(x) = \sup_{Q\ni x}\prod_{i=1}^{m}\frac{1}{\sigma_i(Q)}\int_Q |f_i|.
$$

We will follow the method of the proof of Buckley's theorem given in [14]. By (2.1), without loss of generality we may assume that the maximal operators considered below are dyadic. We get

$$
\prod_{i=1}^{m}\frac{1}{|Q|}\int_Q |f_i| = A_{\vec{P}}(\vec{w}; Q)^{\frac{m}{r-1}}\left(\frac{|Q|}{\nu_{\vec{w}}(Q)}\left(\prod_{i=1}^{m}\frac{1}{\sigma_i(Q)}\int_Q |f_i|\right)^{\frac{r-1}{m}}\right)^{\frac{m}{r-1}}.
$$

Hence,

$$
\mathcal{M}(\vec{f})(x) \leq [\vec{w}]_{A_{\vec{P}}}^{\frac{m}{r-1}}M_{\nu_{\vec{w}}}\left(\mathcal{M}_{\vec{\sigma}}(\vec{f})^{\frac{r-1}{m}}\nu_{\vec{w}}^{-1}\right)(x)^{\frac{m}{r-1}}.
$$

From this, using Hölder's inequality and the boundedness of the weighted dyadic maximal operator with the implicit constant independent of the weight, we obtain

$$
\begin{aligned}
\|\mathcal{M}(\vec{f})\|_{L^p(\nu_{\vec{w}})} &\leq [\vec{w}]_{A_{\vec{P}}}^{\frac{m}{r-1}} \|M_{\nu_{\vec{w}}}\big(\mathcal{M}_{\vec{\sigma}}(\vec{f})^{\frac{r-1}{m}}\nu_{\vec{w}}^{-1}\big)\|_{L^{r'}(\nu_{\vec{w}})}^{\frac{m}{r-1}} \\
&\leq C[\vec{w}]_{A_{\vec{P}}}^{\frac{m}{r-1}} \|\mathcal{M}_{\vec{\sigma}}(\vec{f})\|_{L^{r/m}(\nu_{\vec{w}}^{1-r'})} \\
&\leq C[\vec{w}]_{A_{\vec{P}}}^{\frac{m}{r-1}} \prod_{i=1}^{m} \|M_{\sigma_i}(f_i\sigma_i^{-1})\|_{L^r(\sigma_i)} \\
&\leq C[\vec{w}]_{A_{\vec{P}}}^{\frac{m}{r-1}} \prod_{i=1}^{m} \|f_i\|_{L^r(w_i)}.
\end{aligned}
$$

This proves that $\alpha \leq \frac{m}{r-1}$. But if $p_1 = p_2 = \cdots = p_m = r$, then $\frac{m}{mp-1} = \frac{m}{r-1}$. Hence, using part (i), we get that $\alpha = \frac{m}{r-1}$. $\qquad\square$

Proof of Theorem 1.3. We start exactly as in the proof of Theorem 1.1. It suffices to prove the main result for $\mathcal{M}^d$. Let $\Omega_k = \{x \in \mathbb{R}^n : \mathcal{M}^d(\vec{f})(x) > a^k\} = \cup_j Q_j^k$, where $a = 2^{m(n+1)}$. Then

$$
\int_{\mathbb{R}^n} \mathcal{M}^d(\vec{f})^p\, u\, dx \leq a^p \sum_{k,j} \left(\prod_{i=1}^m \frac{1}{|Q_j^k|} \int_{Q_j^k} |f_i| v_i^{\frac{1}{p_i}} v_i^{-\frac{1}{p_i}}\, dy_i \right)^p u(Q_j^k).
$$

By the generalized Hölder inequality (2.4),

$$
\frac{1}{|Q_j^k|} \int_{Q_j^k} |f_i| v_i^{\frac{1}{p_i}} v_i^{-\frac{1}{p_i}}\, dy_i \leq \|f_i v_i^{\frac{1}{p_i}}\|_{X_i',Q_j^k} \|v_i^{-\frac{1}{p_i}}\|_{X_i,Q_j^k}.
$$

Combining this with the previous estimate, using the properties of the sets E_j^k associated with $\{Q_j^k\}$, and applying Hölder's inequality, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^n} \mathcal{M}^d(\vec{f})^p\, u\, dx &\leq a^p \sum_{k,j} \left(\prod_{i=1}^m \|f_i v_i^{\frac{1}{p_i}}\|_{X_i',Q_j^k} \|v_i^{-\frac{1}{p_i}}\|_{X_i,Q_j^k} \right)^p \frac{u(Q_j^k)}{|Q_j^k|} |Q_j^k| \\
&\leq 2a^p K^p \sum_{k,j} \left(\prod_{i=1}^m \|f_i v_i^{\frac{1}{p_i}}\|_{X_i',Q_j^k} \right)^p |E_j^k| \leq 2a^p K^p \left\| \prod_{i=1}^m M_{X_i'}(f_i v_i^{1/p_i}) \right\|_{L^p}^p \\
&\leq 2a^p K^p \prod_{i=1}^m \|M_{X_i'}(f_i v_i^{1/p_i})\|_{L^{p_i}}^p \leq 2a^p K^p \prod_{i=1}^m \|M_{X_i'}\|_{L^{p_i}}^p \|f_i\|_{L^{p_i}(v_i)}^p,
\end{aligned}
$$

which completes the proof. $\qquad\square$

4. PROOF OF THEOREM 1.4

Theorem 3.2 from [18] says that $M_\delta^\#(T(\vec{f}))(x) \leq C\mathcal{M}(\vec{f})(x)$, where $M^\#$ is the standard Fefferman-Stein sharp function and $M_\delta^\#(f) = M^\#(|f|^\delta)^{1/\delta}$. A simple examination of the proof of this result yields the following local mean oscillation estimate for $T(\vec{f})$.

Proposition 4.1. *For any cube $Q \subset \mathbb{R}^n$,*

(4.1) $$\omega_\lambda(T(\vec{f}); Q) \leq c(T, \lambda, n) \sum_{l=0}^{\infty} \frac{1}{2^{l\epsilon}} \prod_{i=1}^{m} \left(\frac{1}{|2^l Q|} \int_{2^l Q} |f_i(y)| dy \right).$$

We turn now to the proof of Theorem 1.4. Combining Proposition 4.1 and Theorem 2.3 with $Q_0 \in \mathcal{D}$, we get that there exists a sparse family $\mathcal{S} = \{Q_j^k\} \in \mathcal{D}$ such that for a.e. $x \in Q_0$,

(4.2) $$|T(\vec{f})(x) - m_{Q_0}(T(\vec{f}))| \leq c\Big(\mathcal{M}(\vec{f})(x) + \sum_{l=0}^{\infty} \frac{1}{2^{l\epsilon}} \mathcal{T}_{\mathcal{S},l}(|\vec{f}|)(x)\Big),$$

where $c = c(n, T)$ and

$$\mathcal{T}_{\mathcal{S},l}(\vec{f})(x) = \sum_{j,k} \Big(\prod_{i=1}^{m} (f_i)_{2^l Q_j^k} \Big) \chi_{Q_j^k}(x).$$

By the weak type property of the m-linear Calderón-Zygmund operators (2.2), assuming, for instance, that each f_i is bounded and with compact support, we get $(T(\vec{f}))^*(+\infty) = 0$. Hence, it follows from (2.3) that $|m_Q(T(\vec{f}))| \to 0$ as $|Q| \to \infty$. Therefore, letting Q_0 to anyone of 2^n quadrants and using Fatou's lemma and (4.2), we obtain

$$\|T(\vec{f})\|_X \leq c(n, T) \Big(\|\mathcal{M}(\vec{f})\|_X + \sum_{l=0}^{\infty} \frac{1}{2^{l\epsilon}} \sup_{\mathcal{S} \in \mathcal{D}} \|\mathcal{T}_{\mathcal{S},l}(\vec{f})\|_X \Big).$$

By Lemma 2.2,
$$\|\mathcal{M}(\vec{f})\|_X \leq c(m, n) \sup_{\mathscr{D}, \mathcal{S}} \|\mathcal{A}_{\mathscr{D}, \mathcal{S}} |\vec{f}|\|_X.$$

Our goal now is to show that

(4.3) $$\sup_{\mathcal{S} \in \mathcal{D}} \|\mathcal{T}_{\mathcal{S},l}(\vec{f})\|_X \leq c(m, n) l \sup_{\mathscr{D}, \mathcal{S}} \|\mathcal{A}_{\mathscr{D}, \mathcal{S}} |\vec{f}|\|_X.$$

This estimate along with the two previous ones would complete the proof.

4.1. **Several auxiliary operators.** Assume that $f_i \geq 0$. Fix $\mathcal{S} = \{Q_j^k\} \in \mathcal{D}$. Applying Proposition 2.1, we can decompose the cubes Q_j^k into 2^n disjoint families F_α such that for any $Q_j^k \in F_\alpha$ there exists a cube $P_{j,k}^{l,\alpha} \in \mathscr{D}_\alpha$ such that $2^l Q_j^k \subset P_{j,k}^{l,\alpha}$ and $\ell_{P_{j,k}^{l,\alpha}} \leq 6\ell_{2^l Q_j^k}$. Hence,

(4.4) $$\mathcal{T}_{\mathcal{S},l}(\vec{f})(x) \leq 6^{nm} \sum_{\alpha=1}^{2^n} \sum_{j,k: Q_j^k \in F_\alpha} \Big(\prod_{i=1}^{m} (f_i)_{P_{j,k}^{l,\alpha}} \Big) \chi_{Q_j^k}(x).$$

Denote

$$\mathcal{T}_{l,\alpha}(\vec{f})(x) = \sum_{j,k} \Big(\prod_{i=1}^{m} (f_i)_{P_{j,k}^{l,\alpha}} \Big) \chi_{Q_j^k}(x).$$

We shall also need the following auxiliary operator

$$\mathcal{M}_{l,\alpha}(\vec{f}_{1,\ldots,m-1}, g)(x) = \sum_{j,k} \Big(\prod_{i=1}^{m-1} (f_i)_{P_{j,k}^{l,\alpha}} \Big) \Big(\frac{1}{|P_{j,k}^{l,\alpha}|} \int_{Q_j^k} g \Big) \chi_{P_{j,k}^{l,\alpha}}(x).$$

This object appears naturally in the following duality relation:

$$(4.5) \qquad \int_{\mathbb{R}^n} \mathcal{T}_{l,\alpha}(\vec{f}) g \, dx = \int_{\mathbb{R}^n} \mathcal{M}_{l,\alpha}(\vec{f}_{1,\ldots,m-1}, g) f_m \, dx.$$

Lemma 4.2. *For any cube $Q \in \mathcal{D}_\alpha$,*

$$\omega_\lambda(\mathcal{M}_{l,\alpha}(\vec{f}_{1,\ldots,m-1}, g); Q) \le c(\lambda, m, n) l g_Q \prod_{i=1}^{m-1} (f_i)_Q.$$

Proof. Let $Q \in \mathcal{D}_\alpha$ and let $x \in Q$. We have

$$\mathcal{M}_{l,\alpha}(\vec{f}_{1,\ldots,m-1}, g)(x) = \sum_{k,j: P_{j,k}^{m,\alpha} \subset Q} + \sum_{k,j: Q \subseteq P_{j,k}^{m,\alpha}}.$$

The second sum is a constant (denote it by c) for $x \in Q$, while the first sum involves only the functions f_i which are supported in Q. We get the following simple estimate:

$$(4.6) \qquad |\mathcal{M}_{l,\alpha}(\vec{f}_{1,\ldots,m-1}, g) - c|\chi_Q(x) \le \prod_{i=1}^{m-1} M(f_i \chi_Q)(x) \mathcal{T}_l(g \chi_Q)(x),$$

where

$$\mathcal{T}_l g(x) = \sum_{j,k} \Big(\frac{1}{|P_{j,k}^{m,\alpha}|} \int_{Q_j^k} g \Big) \chi_{P_{j,k}^{m,\alpha}}(x).$$

It was proved in [17, Lemma 3.2] that $\|\mathcal{T}_l g\|_{L^{1,\infty}} \le c(n) l \|g\|_{L^1}$. Using this estimate, the weak type $(1,1)$ of M, and reiterating the well known property of rearrangements, $(fg)^*(t) \le f^*(t/2) g^*(t/2)$, $t > 0$, we get using (4.6)

$$\omega_\lambda(\mathcal{M}_{l,\alpha}(\vec{f}_{1,\ldots,m-1}, g); Q) \le \Big(\prod_{i=1}^{m-1} M(f_i \chi_Q) \mathcal{T}_l(g \chi_Q) \Big)^* (\lambda |Q|)$$

$$\le \prod_{i=1}^{m-1} \big(M(f_i \chi_Q) \big)^* \big(\lambda |Q|/2^i \big) \big(\mathcal{T}_l(g \chi_Q) \big)^* \big(\lambda |Q|/2^{m-1} \big)$$

$$\le c(\lambda, m, n) l g_Q \prod_{i=1}^{m-1} (f_i)_Q,$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.2. **Proof of (4.3).** By (4.4) it is enough to prove (4.3) with $\mathcal{T}_{l,\alpha}(\vec{f})$, for each $\alpha = 1, \ldots, 2^n$, instead of $\mathcal{T}_{\mathcal{S},l}(\vec{f})$ on the left-hand side. By the standard limiting argument one can assume that the sum defining $\mathcal{T}_{l,\alpha}(\vec{f})$ is finite. Then the sum defining the corresponding operator $\mathscr{M}_{l,\alpha}(\vec{f}_{1,\ldots,m-1}, g)$ in (4.5) will be finite too. This means that the support of $\mathscr{M}_{l,\alpha}(\vec{f}_{1,\ldots,m-1}, g)$ is compact. One can cover it by at most 2^n cubes $Q_\nu \in \mathscr{D}_\alpha$ such that

$$m_{\mathscr{M}_{l,\alpha}(\vec{f}_{1,\ldots,m-1},g)}(Q_\nu) = 0, \qquad \nu = 1, \cdots, 2^n.$$

Applying Theorem 2.3 along with Lemma 4.2, we get that there exists a sparse family $\mathcal{S}_\alpha \in \mathscr{D}_\alpha(Q_\nu)$ such that for a.e. $x \in Q_\nu$,

$$\mathscr{M}_{l,\alpha}(\vec{f}_{1,\ldots,m-1}, g)(x)$$
$$\leq c(m,n)l\Big(\mathcal{M}^{\mathscr{D}_\alpha}(\vec{f}_{1,\ldots,m-1}, g)(x) + \sum_{Q_j^k \in \mathcal{S}_\alpha} \Big(\prod_{i=1}^{m-1}(f_i)_{Q_j^k}\Big)g_{Q_j^k}\chi_{Q_j^k}(x)\Big),$$

where

$$\mathcal{M}^{\mathscr{D}_\alpha}(\vec{f}_{1,\ldots,m-1}, g)(x) = \sup_{x \in Q \in \mathscr{D}_\alpha} \prod_{i=1}^{m-1}(f_i)_Q g_Q.$$

Applying to this maximal operator the same argument as in the proof of Lemma 2.2 and combining with the previous estimate, we get that there exists two sparse families $\mathcal{S}_{\alpha,1}$ and $\mathcal{S}_{\alpha,2}$ from $\mathscr{D}_\alpha$ such that for a.e. $x \in Q_\nu$,

$$\mathscr{M}_{l,\alpha}(\vec{f}_{1,\ldots,m-1}, g)(x) \leq c(m,n)l\sum_{\kappa=1}^{2}\sum_{Q_j^k \in \mathcal{S}_{\alpha,\kappa}}\Big(\prod_{i=1}^{m-1}(f_i)_{Q_j^k}\Big)g_{Q_j^k}\chi_{Q_j^k}(x).$$

Hence, by Hölder's inequality (2.4),

$$\int_{Q_\nu}\mathscr{M}_{l,\alpha}(\vec{f}_{1,\ldots,m-1}, g)f_m\,dx \leq c(m,n)l\sum_{\kappa=1}^{2}\int_{\mathbb{R}^n}\mathcal{A}_{\mathscr{D}_\alpha,\mathcal{S}_{\alpha,\kappa}}(\vec{f})g\,dx$$
$$\leq 2c(m,n)l\sup_{\mathscr{D},\mathcal{S}}\|\mathcal{A}_{\mathscr{D},\mathcal{S}}(\vec{f})\|_X\|g\|_{X'}.$$

Summing up over Q_ν and using (4.5), we get

$$\int_{\mathbb{R}^n}\mathcal{T}_{l,\alpha}(\vec{f})g\,dx \leq 2^{n+1}c(m,n)l\sup_{\mathscr{D},\mathcal{S}}\|\mathcal{A}_{\mathscr{D},\mathcal{S}}(\vec{f})\|_X\|g\|_{X'}.$$

By (2.5), taking here the supremum over $g \geq 0$ with $\|g\|_{X'} = 1$ gives (4.3) for $\mathcal{T}_{l,\alpha}(\vec{f})$, and therefore the proof is complete.

5. Proof of Theorem 1.5

First, we apply Theorem 1.4 with $X = L^p(\nu_{\vec{w}})$ (observe that $p = \frac{m+1}{m}$). Fix $S \in \mathscr{D}$. Assume that $f_i \geq 0$. By duality

$$\|\mathcal{A}_{\mathscr{D},S}(\vec{f})\|_{L^p(\nu_{\vec{w}})} = \sup_{\|g\|_{L^{p'}\left(\nu_{\vec{w}}^{-1/(p-1)}\right)=1}} \sum_{j,k} \prod_{i=1}^m (f_i)_{Q_j^k} \int_{Q_j^k} g.$$

Observe that by our choice of p_i we have $p/p_i' = 1$. Denote

$$A_{\vec{P}}(\vec{w}; Q) = \Big(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}\Big) \prod_{i=1}^m \Big(\frac{1}{|Q|}\int_Q \sigma_i\Big).$$

We have

$$\sum_{j,k} \prod_{i=1}^m (f_i)_{Q_j^k} \int_{Q_j^k} g$$

$$= \sum_{j,k} A_{\vec{P}}(\vec{w}; Q_j^k)\Big(\prod_{i=1}^m \frac{1}{\sigma_i(Q_j^k)}\int_{Q_j^k} f_i\Big)\Big(\frac{1}{\nu_{\vec{P}}(Q_j^k)}\int_{Q_j^k} g\Big)|Q_j^k|$$

$$\leq 2[\vec{w}]_{A_{\vec{P}}} \sum_{j,k} \int_{E_j^k} \prod_{i=1}^m M_{\sigma_i}^{\mathscr{D}}(f_i\sigma_i^{-1}) M_{\nu_{\vec{w}}}^{\mathscr{D}}(g\nu_{\vec{w}}^{-1})dx$$

$$\leq 2[\vec{w}]_{A_{\vec{P}}} \int_{\mathbb{R}^n} \prod_{i=1}^m M_{\sigma_i}^{\mathscr{D}}(f_i\sigma_i^{-1}) M_{\nu_{\vec{w}}}^{\mathscr{D}}(g\nu_{\vec{w}}^{-1})dx.$$

Now we apply Hölder's inequality:

$$\int_{\mathbb{R}^n} \prod_{i=1}^m M_{\sigma_i}^{\mathscr{D}}(f_i\sigma_i^{-1}) M_{\nu_{\vec{w}}}^{\mathscr{D}}(g\nu_{\vec{w}}^{-1})dx$$

$$\leq \|\prod_{i=1}^m M_{\sigma_i}^{\mathscr{D}}(f_i\sigma_i^{-1})\|_{L^p(\nu_{\vec{w}}^{-(p-1)})}\|M_{\nu_{\vec{w}}}^{\mathscr{D}}(g\nu_{\vec{w}}^{-1})\|_{L^{p'}(\nu_{\vec{w}})}.$$

First,

$$\|M_{\nu_{\vec{w}}}^{\mathscr{D}}(g\nu_{\vec{w}}^{-1})\|_{L^{p'}(\nu_{\vec{w}})} \leq c\|g\|_{L^{p'}\left(\nu_{\vec{w}}^{-1/(p-1)}\right)} = c.$$

Second, we apply Hölder's inequality with p_i/p and use that $p - 1 = \frac{1}{p_i-1}$. We get

$$\|\prod_{i=1}^m M_{\sigma_i}^{\mathscr{D}}(f_i\sigma_i^{-1})\|_{L^p(\nu_{\vec{w}}^{-(p-1)})} \leq \prod_{i=1}^m \|M_{\sigma_i}^{\mathscr{D}}(f_i\sigma_i^{-1})\|_{L^{p_i}(\sigma_i)} \leq c\prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)},$$

and we are done.

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